FAQs & their solutions for Module 8:
Angular Momentum-II

Question1: The spin angular momentum operator for electron is given by

$$s_x = \frac{1}{2}\hbar\sigma_x \;,\quad s_y = \frac{1}{2}\hbar\sigma_y \;,\; s_z = \frac{1}{2}\hbar\sigma_z \qquad (1)$$

Where σ_x , σ_y and σ_z are Pauli spin matrices and are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Write the eigenvalues and eigenvectors of s_x , s_y and s_z.

Solution1: We first determine the eigenvalues of the σ_x matrix which are determined from the following equation

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

or

$$\lambda^2 - 1 = 0`$$

implying

$$\lambda = \pm 1 \qquad (2)$$

Thus the eigenvalues of the σ_x matrix are $\pm$ 1 and therefore the eigenvalues of s_x are $\pm\frac{1}{2}\hbar$. Thus if we measure s_x [i.e., the x component of the spin angular momentum of a spin $\frac{1}{2}$ particle like electron, proton or neutron] then we will get only one of the two possible (eigen) values $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$. The corresponding eigenfunctions are easy to determine; e.g., for the eigenvalue $+\frac{1}{2}\hbar$, the eigenvalue equation is written as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +1\begin{pmatrix} a \\ b \end{pmatrix}$$

giving

$$b = a$$

Thus the eigenfunction is given by

$$a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If we normalize the eigenvector we will get

$$\left|\hat{\mathbf{x}}\uparrow\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{3}$$

which is usually referred to as the "*x*-up" state. Similarly

$$\left|\hat{\mathbf{x}}\downarrow\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{4}$$

represents the normalized eigenvector corresponding to the eigenvalue $-\frac{1}{2}\hbar$ (of s_x) and is usually referred to as the "*x*-down" state.

Since σ_z is a diagonal matrix, the eigenvalues of σ_z are just +1 and –1 implying that the eigenvalues of s_z are $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$. The corresponding (normalized) eigenfunction are easy to determine and are given by

$$\left|\hat{\mathbf{z}}\uparrow\right\rangle = \begin{pmatrix}1\\0\end{pmatrix} \tag{5}$$

and

$$\left|\hat{\mathbf{z}}\downarrow\right\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{6}$$

corresponding to the "z-up" state (eigenvalue $+\frac{1}{2}\hbar$ of s_z) and the "z-down" state (eigenvalue $-\frac{1}{2}\hbar$ of s_z) respectively. Finally the eigenvalues of σ_y are determined from the following equation

$$\begin{vmatrix}-\lambda & -i\\ +i & -\lambda\end{vmatrix} = 0$$

or

$$\lambda^2 - 1 = 0`$$

implying

$$\lambda = \pm 1 \tag{7}$$

Thus the eigenvalues of s_y are (again) $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$. Corresponding to the eigenvalue $-\frac{1}{2}\hbar$, the eigen function is determined from the equation

$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = -\begin{pmatrix}a\\b\end{pmatrix}$$

or

$$-ib = -a \Rightarrow b = -ia$$

Thus

$$\left|\hat{\mathbf{y}}\downarrow\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \tag{8}$$

would represent the normalized "*y*-down" state. Similarly

$$\left|\hat{\mathbf{y}}\uparrow\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \tag{9}$$

would represent the normalized "*y*-up" state.

Question2: A spin half particle is in the "z-up" state. On that particle, if we make a measurement of s_x then what are the values that we will obtain and what will be their probabilities.

Solution2: The spin half particle is in the "z-up" state. On that particle, if we make a measurement of s_x then we will get one of the two eigenvalues of s_x. In order to determine their probabilities we have to express the (normalized) "z-up" state as a linear combination of the (normalized) "*x*-up" and "*x*-down" states:

$$|\hat{\mathbf{z}}\uparrow>=\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{\sqrt{2}}|\hat{\mathbf{x}}\uparrow>+\frac{1}{\sqrt{2}}|\hat{\mathbf{x}}\downarrow> \tag{10}$$

Thus, if we make a measurement of s_x then the probability of obtaining a "*x*-up" state [i.e., the probability of obtaining the eigenvalue $+\frac{1}{2}\hbar$ for s_x] is $\frac{1}{2}$ and the probability of obtaining a "*x*-down" state is also $\frac{1}{2}$.

Question3: The magnetic moment of the neutral Ag-atom is the same as that of an electron and is given by

$$\boldsymbol{\mu} \approx -\frac{q}{m}\,\mathbf{s} \tag{11}$$

where$q$ and m represent the charge and mass of the electron and

$$s_x = \frac{1}{2}\hbar\sigma_x \;\;,\;\; s_y = \frac{1}{2}\hbar\sigma_y \;\;,\;\; s_z = \frac{1}{2}\hbar\sigma_z \tag{12}$$

Such a particle is placed in a static magnetic field given by

$$\mathbf{B} = B_0\ \hat{\mathbf{z}} \tag{13}$$

Obtain the eigenvalues and eigenfunctionsof the energy associated with magnetic field.

Solution3: The magnetic moment of the neutral Ag-atom is the same as that of an electron and is given by

$$\boldsymbol{\mu} \approx -\frac{q}{m}\,\mathbf{s} \tag{14}$$

where
$$s_x = \frac{1}{2}\hbar\sigma_x \;\;,\;\; s_y = \frac{1}{2}\hbar\sigma_y \;\;,\;\; s_z = \frac{1}{2}\hbar\sigma_z \tag{15}$$

where σ_x , σ_y and σ_z are Pauli spin matrices. If such a particle is placed in a static magnetic field given by

$$\mathbf{B} = B_0\;\hat{\mathbf{z}} \tag{16}$$

then the potential energy associated with magnetic field would be given by

$$H_0 = -\boldsymbol{\mu}\,.\mathbf{B} = \frac{1}{2}\hbar\omega_0\sigma_z \tag{17}$$

where

$$\omega_0 \equiv \frac{qB_0}{m} = \frac{2\mu_B B_0}{\hbar} \tag{18}$$

and

$$\mu_B = \frac{q\hbar}{2m} \simeq 9.274\times10^{-24}\;\;\text{J/T} \tag{19}$$

represents the Bohr magneton. Since the eigenvalues of σ_z are +1 and -1, the solution of the eigenvalue equation

$$H_0\,|\,n>=E_n\,|\,n>\,;\;\; n=1,2 \tag{20}$$

would be given by

$$E_1 = \frac{1}{2}\,\hbar\omega_0 \Leftrightarrow\; |1> \;=\; |\,\hat{\mathbf{z}}\uparrow> = \begin{pmatrix}1\\0\end{pmatrix} \tag{21}$$

$$E_2 = -\frac{1}{2}\,\hbar\omega_0 \Leftrightarrow |\,2> \;= |\,\hat{\mathbf{z}}\downarrow> = \begin{pmatrix}0\\1\end{pmatrix} \tag{22}$$

Question4: Write the most general solution of the time dependent Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}|\,\Psi(t)> \;= H_0\,|\,\Psi(t)> \quad (23)$$

where

$$H_0 = \frac{1}{2}\hbar\omega_0\sigma_z$$

Solution4 : The most general solution of the time dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} | \Psi(t) > \; = H_0 | \Psi(t) > = \frac{1}{2} \hbar \omega_0 \sigma_z | \Psi(t) > \tag{24}$$

would be

$$\begin{aligned} | \Psi(t) > \; &= \sum_{n=1}^{2} C_n \; e^{-iE_n t/\hbar} \; | n > \\ &= C_1 \, e^{-i\omega_0 t/2} \; |1> + C_2 \, e^{+i\omega_0 t/2} \; |2> \end{aligned} \tag{25}$$

where

$$E_1 = \frac{1}{2} \hbar\omega_0 \Leftrightarrow |1> \; = \; | \hat{\mathbf{z}} \uparrow > = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{26}$$

$$E_2 = -\frac{1}{2} \hbar\omega_0 \Leftrightarrow |2> \; = | \hat{\mathbf{z}} \downarrow > = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{27}$$

Further, the coefficients C_1 and C_2 are to be determined from the knowledge of $|\Psi(t = 0) >$:

$$C_1 = \langle 1 | \Psi(0) \rangle$$

and

$$C_1 = \langle 2 | \Psi(0) \rangle$$

Of course, if the system is initially in the $| \; \hat{\mathbf{z}} \uparrow >$ or $| \; \hat{\mathbf{z}} \downarrow >$ states, then it will remain in those states for all times to come; these are the *stationary states* of the problem.

Question5: In continuation of the previous problem, we assume that at $t = 0$, the atom is in the $| \hat{x} \uparrow \rangle$ state. Obtain the time evolution of the state and calculate the expectation values of s_x, s_y and s_z.

Solution5:

At $t = 0$, the atom is in the $| \; \hat{\mathbf{x}} \uparrow >$ state. Since

$$| \hat{\mathbf{x}} \uparrow \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

we readily get

$$C_1 = < \hat{\mathbf{z}} \uparrow | \hat{\mathbf{x}} \uparrow > \; = \frac{1}{\sqrt{2}} (1 \quad 0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

Similarly $C_2 = 1/\sqrt{2}$. Thus

$$\left|\Psi(t)\right\rangle = \frac{1}{\sqrt{2}} e^{-i\theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} e^{i\theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

or

$$\left|\Psi(t)\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta} \\ e^{i\theta} \end{pmatrix} \tag{28}$$

where

$$\theta = \frac{1}{2}\omega_0 \, t \tag{29}$$

Equation (21)(28) describes the time evolution of the state. Further,

$$\begin{aligned} \langle s_x \rangle = & \ \frac{1}{2}\hbar \ \langle \Psi(t) | \ \sigma_x | \Psi(t) \rangle \\ = & \ \frac{1}{4}\hbar \begin{pmatrix} e^{i\theta} & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} \\ e^{i\theta} \end{pmatrix} \end{aligned}$$

or

$$\langle s_x \rangle = \frac{1}{2}\hbar \cos\omega_0 t \tag{30}$$

Similarly

$$\langle s_y \rangle = \frac{1}{2} \ \hbar \sin\omega_0 t \tag{31}$$

and

$$\langle s_z \rangle = 0 \tag{32}$$

The above equations physically imply that the direction of the spin angular momentum vector rotates about the *z*-axis with angular velocity ω_0 .

<u>Question6:</u> Next consider the more general case when

$$\left|\Psi(0)\right\rangle = \cos\frac{\phi}{2}\left|\hat{\mathbf{z}}\uparrow\right\rangle + \sin\frac{\phi}{2}\left|\hat{\mathbf{z}}\downarrow\right\rangle$$

Obtain the time evolution of the state.

<u>Solution6:</u>

We have

$$\left|\Psi(0)\right\rangle = \cos\frac{\phi}{2}\left|\hat{\mathbf{z}}\uparrow\right\rangle + \sin\frac{\phi}{2}\left|\hat{\mathbf{z}}\downarrow\right\rangle \tag{33}$$

[when $\phi = \pi/2$, we obtain the results of the previous problem]. Obviously

$$C_1 = \cos\frac{\phi}{2} \ \text{ and } \ C_2 = \sin\frac{\phi}{2}$$

so that

$$
\begin{aligned}
\left|\Psi(t)\right\rangle &= \cos\frac{\phi}{2}e^{-i\theta}\left|\hat{\mathbf{z}}\uparrow\right\rangle + \sin\frac{\phi}{2}e^{+i\theta}\left|\hat{\mathbf{z}}\downarrow\right\rangle \\
&= \begin{pmatrix} \cos\frac{\phi}{2}\ e^{-i\theta} \\ \sin\frac{\phi}{2}\ e^{+i\theta} \end{pmatrix}
\end{aligned}
$$

Thus

$$\left\langle s_x \right\rangle = \frac{1}{2}\hbar\ \left\langle \Psi(t)\right|\sigma_x\left|\Psi(t)\right\rangle$$

or

$$\left\langle s_x \right\rangle = \frac{1}{2}\ \hbar \sin\phi \cos\omega_0 t \tag{34}$$

Similarly

$$\left\langle s_y \right\rangle = \frac{1}{2}\ \hbar \sin\phi \sin\omega_0 t \tag{35}$$

and

$$\left\langle s_z \right\rangle = \frac{1}{2}\ \hbar\ \cos\phi \tag{36}$$